Area Under a Curve

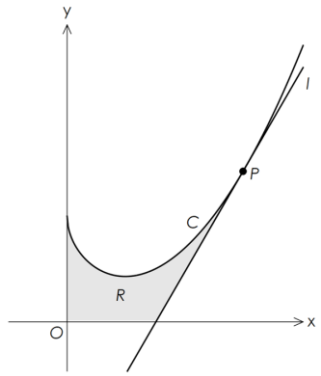

Figure 1

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

Figure 1 shows a sketch of part of the curve C with equation

$$y = \frac{1}{2}x^2 - 3\sqrt{x} + 5$$

The point *P* lies on C and has x coordinate 4

The line *l* is the tangent to C at *P*.

a. Show that *l* has equation

$$13x - 4y - 24 = 0$$

(5 marks)

The region *R*, shown shaded in Figure 1, is bounded by the y-axis, the curve C, the line *l* and the x-axis.

b. Find the exact area of *R*.

(5 marks)

a. Begin by changing the root in the equation of curve C to a power

$$y = \frac{1}{2}x^2 - 3\sqrt{x} + 5 \Rightarrow y = \frac{1}{2}x^2 - 3x^{\frac{1}{2}} + 5$$

1 mark

and then differentiate

$$y = \frac{1}{2}x^2 - 3x^{\frac{1}{2}} + 5 \Rightarrow \frac{dy}{dx} = x - \frac{3}{2}x^{-\frac{1}{2}}$$

1 mark

Substitute $x = 4$ into the equation for the curve C to find the corresponding y coordinate, and into the differentiated equation to find the gradient of the tangent to C at P.

$$\frac{1}{2} \times 4^2 - 3 \times \sqrt{4} + 5 = 7$$ 1 mark

$$4 - \frac{3}{2} \times \frac{1}{\sqrt{4}} = \frac{13}{4}$$ 1 mark

$$y - y_1 = m(x - x_1)$$

$$y - 7 = \frac{13}{4}(x - 4)$$

$$y - 7 = \frac{13}{4}x - 13$$

$$y = \frac{13}{4}x - 6$$

$$4y = 13x - 24$$

$$13x - 4y - 24 = 0$$ 1 mark

b. To find the area of the region R, integrate between $x = 0$ and $x = 4$, and then subtract the area of the triangle bound by the line l, the x-axis and the line $x = 4$.
The line l crosses the x-axis when $y = 0$:

$$13x - 0 - 24 = 0$$

$$x = \frac{24}{13}$$ 1 mark

So the area of the triangle is:

$$\frac{1}{2} \times \left(4 - \frac{24}{13}\right) \times 7 = \frac{98}{13}$$

Area under the curve, bound by the x-axis, the y-axis and the line $x = 4$ is

$$\int_0^4 \left(\frac{1}{2}x^2 - 3x^{\frac{1}{2}} + 5\right) dx$$ 1 mark

$$= \left[\frac{1}{6}x^3 - 2x^{\frac{3}{2}} + 5x\right]_0^4$$ 1 mark

$$= \frac{1}{6} \times 64 - 2 \times 8 + 5 \times 4$$

$$= \frac{32}{3} + 4$$

$$= \frac{44}{3}$$

Area of region R is

$$\frac{44}{3} - \frac{98}{13}$$ 1 mark

$$= \frac{278}{39}$$ 1 mark